THE BOUNDED SPHERICAL FUNCTIONS ON THE CARTAN MOTION GROUP

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ABSTRACT. The bounded spherical functions are determined for a complex Cartan motion group.

1. INTRODUCTION

Consider a symmetric space $X = G/K$ of noncompact type, G being a connected noncompact semisimple Lie group with finite center and K a maximal compact subgroup. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition, $\mathfrak{p}$ being the orthocomplement of $\mathfrak{k}$ relative to the Killing form $B(= \langle\ ,\ \rangle)$ of $\mathfrak{g}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, Σ the set of root of $\mathfrak{g}$ relative to $\mathfrak{a}$, $\mathfrak{a}^+$ a fixed Weyl chamber and Σ^+ the set of roots α positive on $\mathfrak{a}^+$. Let ρ denote the half sum of the $\alpha \in \Sigma^+$ with multiplicity. The spherical functions on X (and G) are by definition the K-invariant joint eigenfunctions of the elements in $\mathbb{D}(X)$, the algebra of G-invariant differential operators on X. By Harish-Chandra's result [HC58] the spherical functions on X are given by

$$\phi_\lambda(gK) = \int_K e^{(i\lambda-\rho)(H(gK))}\, dk, \quad \phi(eK) = 1, \tag{1.1}$$

where $\exp H(g)$ is the A factor in the Iwasawa decomposition $G = KAN$ (N nilpotent) and λ ranges over the space $\mathfrak{a}_c^*$ of complex-valued linear functions on $\mathfrak{a}$. Also, $\phi_\lambda \equiv \phi_\mu$ if and only if the elements $\lambda, \mu \in \mathfrak{a}_c^*$ are conjugate under W.

Let $L^\natural(G)$ denote the (commutative) Banach algebra of K-bi-invariant integrable functions on G. The maximal ideal space of $L^\natural(G)$ is known to consist of the kernels of the spherical transforms

$$f \to \int_G f(g)\phi_{-\lambda}(g)\, dg$$

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for which $\phi_{-\lambda}$ is bounded. These bounded spherical functions were in [HJ69] found to be those ϕ_λ for which λ belongs to the tube $a^* + iC(\rho)$ where $C(\rho)$ is the convex hull of the points $s\rho (s \in W)$.

This result is crucial in proving that the horocycle Radon transform is injective on $L^1(X)$ ([H70], Ch. II).

2. The boundedness criterion.

In this note we deal with the analogous question for the Cartan motion group G_0. This group is defined as the semidirect product of K and $\mathfrak{p}$ with respect to the adjoint action of K on $\mathfrak{p}$. The $X_0 = G_0/K$ is naturally identified with the Euclidean space $\mathfrak{p}$. The element $g_0 = (k, Y)$ actions on $\mathfrak{p}$ by

$$g_0(Y') = Ad(k)Y' + Y \quad k \in K,\ Y, Y' \in \mathfrak{p},$$

so the algebra $\mathbb{D}(X_0)$ of G_0-invariant differential operators on X_0 is identified with the algebra of $Ad(K)-$invariant constant coefficient differential operators on $\mathfrak{p}$. The corresponding spherical functions on X_0 are given by

$$\psi_\lambda(Y) = \int_K e^{i\lambda(Ad(k)Y)}\, dk \quad \lambda \in \mathfrak{a}_c^*, \tag{2.1}$$

and $\psi_\lambda \equiv \psi_\mu$ if and only if λ and μ are $W-$conjugate. See e.g. [H84], IV §4. Again, the maximal ideal space of $L^\natural(G_0)$ is up to $W-$invariance identified with the set of λ in $\mathfrak{a}_c^*$ for which ψ_λ is bounded. Since ρ is related to the curvature of G/K it is natural to expect the bounded ψ_λ to come from replacing $C(\rho)$ by the origin, in other words ψ_λ is would be expected to be bounded if and only if λ is real, that is $\lambda \in \mathfrak{a}^*$.

The bounded criterion in [HJ69] for X relies on Harish-Chandra's expansion for ϕ_λ, combined with the reduction to the boundary components of X. These are certain subsymmetric spaces of X.These tools are not available for X_0 so the "tangent space analysis" in [H80] relies on approximating ψ_λ by ϕ_λ suitably modified. Although several papers ([BC86], [R88], [SØ05]) are directed to asymptotic properties of the function ψ_λ the boundedness question does not seem to be addressed there. In this note we only give a partial solution through the following result.

Theorem 2.1. *Assume the group G complex. The spherical function ψ_λ on G_0 is bounded if and only if λ is real, i.e. $\lambda \in \mathfrak{a}^*$.*

For $\lambda \in \mathfrak{a}_c^*$ let $\lambda = \xi + i\eta$ with $\xi, \eta \in \mathfrak{a}^*$. It remains to prove that if $\lambda_0 = \xi_0 + i\eta_0$ with $\eta_0 \neq 0$ then ψ_{λ_0} is unbounded. For $\lambda \in \mathfrak{a}_c^*$ let $A_\lambda \in \mathfrak{a}_c$ be determined by $\langle A_\lambda, H\rangle = \lambda(H)$ $(H \in \mathfrak{a})$. With $i\lambda_0 = i\xi_0 - \eta_0$ we may by the W-invariance of ψ_λ in λ assume that $-A_{\eta_0} \in \overline{\mathfrak{a}^+}$ (the closure of $\mathfrak{a}^+$.)

Let $U \subset W$ be the subgroup fixing λ_0 and $V \subset W$ the subgroup fixing η_0. Then $U \subset V$ and

$$\psi_{s\xi_0 + i\eta_0} = \psi_{\xi_0 + i\eta_0} \quad \text{for } s \in V. \tag{2.2}$$

In addition we assume that for the lexicographic ordering of $\mathfrak{a}^*$ defined by the simple roots $\alpha_1, \ldots, \alpha_\ell$ we have $\xi_0 \geq s\xi_0$ for $s \in V$.

In particular,

$$\alpha(A_{\xi_0}) \geq 0 \quad \text{for } \alpha \in \Sigma^+ \text{ satisfying } \alpha(A_{\eta_0}) = 0. \tag{2.3}$$

Lemma 2.2. *The subgroup U of W fixing λ_0 is generated by the reflections s_{α_i} where α_i is a simple root vanishing at A_{λ_0}.*

Proof. We first prove that some of the α_i vanishes at A_{λ_0}. The group U is generated by the s_α for which $\alpha > 0$ vanishes on λ_0 ([H78], VII, Theorem 2.15). If α is such then $\alpha(-A_{\eta_0}) = 0$ and since $\alpha = \Sigma_j n_j \alpha_j$ ($n_j \neq 0$ in $\mathbb{Z}^+$) and $\alpha_j(-A_{\eta_0}) \geq 0$ we see that each of these α_j vanishes on $A_{-\eta_0}$. Since $\alpha(A_{\xi_0}) = 0$ and $\alpha_j(A_{\xi_0}) \geq 0$ by (2.3) for each j we deduce $\alpha_j(A_{\xi_0}) = 0$.

Let U' denote the subgroup U generated by those s_{α_i} with α_i vanishing at λ_0. For each $\alpha > 0$ mentioned above we shall prove $\alpha = s\alpha_p$ where $s \in U'$ and α_p is simple and vanishes at A_{λ_0}. We shall prove this by induction on $\sum_i m_i$ if $\alpha = \sum m_i \alpha_i$ ($m_i \neq 0$ in $\mathbb{Z}^+$). The statement is clear if $\sum m_i = 1$ so assume $\sum m_i > 1$. Since $\langle \alpha, \alpha\rangle > 0$ we have $\langle \alpha, \alpha_k\rangle > 0$ for some k among the indices i above. Then $\alpha \neq \alpha_k$ (by $\sum m_i > 1$). Since s_{α_k} permutes the positive roots $\neq \alpha_k$ we have $s_{\alpha_k}\alpha \in \sum^+$ and $s_{\alpha_k}\alpha = \sum_j m'_j \alpha_j (m'_j \in \mathbb{Z}^+)$ and by the choice of $k, \sum m'_j < \sum m_i$. Now $\alpha(A_{i\lambda_0}) = 0$ and $\alpha_i(A_{-\eta_0}) \geq 0$ so for each i in the sum for α above, $\alpha_i(A_{\eta_0}) = 0$. Hence by (2.3) $\alpha_i(A_{\xi_0}) = 0$. In particular $s_{\alpha_k} \in U$. Thus the induction assumption applies to $s_{\alpha_k}\alpha$ giving a $s' \in U'$ for which $s_{\alpha_k}\alpha = s'\alpha_p$. Hence $\alpha = s\alpha_p$ with $s \in U'$. But then $s_\alpha = s s_{\alpha_k} s^{-1}$ proving the lemma. □

Using Harish-Chandra's integral formula [HC57] Theorem 2 we have

$$\psi_\lambda(\exp H) = c_0 \frac{\sum_{s\in W} \epsilon(s) e^{i\langle sA_\lambda, H\rangle}}{\pi(H)\,\pi(A_\lambda)} \qquad \langle H \in \mathfrak{a}\rangle, \tag{2.4}$$

where c_0 is a constant, $\langle\ ,\ \rangle$ the Killing form, $\epsilon(s) = \det s$ and π the product of the positive roots. If η_0 is regular so $-A_{\eta_0} \in \mathfrak{a}^+$ then $V = U = \{e\}$ and $\pi(A_{\lambda_0}) \neq 0$. Fix $H_0 \in \mathfrak{a}^+$ and $\lambda = \lambda_0$ in the sum (2.4). With $H = tH_0 (t > 0)$ the term in (2.4) with $s = e$ will outweigh all the others as $t \to +\infty$ so ψ_λ is unbounded.

We now consider the case $\pi(A_{\lambda_0}) = 0$.

Let π' denote the product of the positive roots $\beta_1, \dots, \beta_r$ vanishing at λ_0 and π'' the product of the remaining positive roots. For $\lambda = \lambda_0$ we want to divide the factor $\pi'(\lambda_0)$ into the numerator of (2.4). We do this by multiplying (2.4) by $\pi'(\lambda)$, then applying the differential operator $\partial(\pi')$ in the variable λ and finally setting $\lambda = \lambda_0$. The theorem then follows from the following lemma.

Lemma 2.3. *Let $\eta_0 \neq 0$. Then the function*

$$\zeta_\lambda(H) = \frac{\sum_{s\in W} \epsilon(s) e^{i\langle sA_\lambda, H\rangle}}{\pi(A_\lambda)}$$

is for the case $\lambda = \lambda_0$ unbounded on $\mathfrak{a}^+$.

Proof. We have

$$\pi'(\lambda)\zeta_\lambda(H) = \frac{1}{\pi''(\lambda)} \sum_{s\in W} \epsilon(s) e^{i\langle sA_\lambda, H\rangle}.$$

Applying $\partial(\pi') = \partial(\beta_1)\dots\partial(\beta_r)$ in λ and putting $\lambda = \lambda_0$ we see that

$$c\ \zeta_{\lambda_0}(H) = \sum_{s\in W} P_s(H) e^{i\langle sA_{\lambda_0}, H\rangle}. \tag{2.5}$$

Here c is a constant and P_s the polynomial

$$P_s(H) = \left[\partial(\pi')_\lambda \left(\epsilon(s)\frac{1}{\pi''(\lambda)} e^{is\lambda(H)}\right)\right]_{\lambda=\lambda_0} e^{-is\lambda_0(H)}$$

whose highest degree term is a constant times

$$\epsilon(s)\frac{1}{\pi''(\lambda_0)}(s\pi')(H). \tag{2.6}$$

We do not need the exact value of c but for $r = 2, 3$, respectively, it equals (with $x_{ij} = \langle \alpha_i, \alpha_j \rangle$)

$$x_{12}^2 + x_{11}x_{22}, \quad x_{11}x_{23}^2 + x_{22}x_{13}^2 + x_{33}x_{12}^2 + x_{11}x_{22}x_{33} + 2x_{12}x_{13}x_{23}.$$

We break the sum (2.5) into two parts, sum over V and sum over $W \backslash V$. For the first we consider Σ_V as $\Sigma_{V/U} \Sigma_U$. Then (2.5) can be written

$$c\,\zeta_{\lambda_0}(H) = e^{-\eta_0(H)} \left[\sum_{V/U} e^{is\xi_0(H)} \sum_{\sigma \in U} P_{s\sigma}(H) \right] + \sum_{W \backslash V} P_s(H) e^{is\lambda_0(H)}. \tag{2.7}$$

We put here $H' = -A_{\eta_0}$, let $H_0 \in \mathfrak{a}^+$ be arbitrary and set $H = tH_0 (t > 0)$. Then the second term in (2.7) equals

$$\sum_{s \notin V} P_s(tH_0) e^{is\xi_0(tH_0)}\, e^{\langle sH', tH_0 \rangle}. \tag{2.8}$$

By a standard property of $\mathfrak{a}^+$ we have

$$\langle H_1, H_2 \rangle \geq \langle sH_1, H_2 \rangle \quad \text{if } H_1, H_2 \in \mathfrak{a}^+$$

so taking limit,

$$\langle sH' - H', H \rangle \leq 0, \quad H \in \mathfrak{a}^+.$$

If $s \notin V$ then $sH' - H' \neq 0$. Thus the map $H \to \langle sH' - H', H \rangle$ is open from $\mathfrak{a}$ to $\mathbb{R}$ mapping $\mathfrak{a}^+$ into $\{t \leq 0\}$, not taking there the boundary value 0. Hence we get

$$\langle H', H_0 \rangle > \langle H', sH_0 \rangle \quad \text{for } s \notin V. \tag{2.9}$$

Equivalently, $\operatorname{dist}(H_0, H') < \operatorname{dist}(H_0, sH')$ for $s \notin V$.

Consider (2.7) with $H = tH_0$. Assume the expression in the bracket has absolute value with $\limsup_{t \to +\infty} \neq 0$. Considering (2.9) the first term in (2.7) would have exponential growth larger than that of each term in (2.8).

Thus $c \neq 0$ and

$$\lim_{t \to +\infty} \zeta_{\lambda_0}(tH_0) = \infty$$

implying Lemma 2.3 in this case.

We shall now exclude the possibility that the quantity in the bracket in (2.7) (with $H = tH_0$) has absolute value with $\limsup_{t\to\infty} = 0$. For this we use the following elementary result of Harish–Chandra [HC58], Corollary of Lemma 56: Let $a_1, \dots a_n$ be nonzero complex numbers and $p_0, \dots p_n$ polynomials with complex coefficients.

Suppose

$$\limsup_{t\to\infty} \left| p_0(t) + \sum_{j=1}^{n} p_j(t) e^{a_j t} \right| \leq a \tag{2.10}$$

for some $a \in \mathbb{R}$. Then p_0 is a constant and $|p_0| \leq a$. This implies the following result.

Let $k_1 \dots k_n \in \mathbb{R}$ be different and $p_1, \dots, p_n$ polynomials. If

$$\limsup_{t\to+\infty} \left| \sum_{1}^{n} e^{ik_r t} p_r(t) \right| = a < \infty \tag{2.11}$$

then each p_r is constant. If $a = 0$ then each $p_r = 0$. This follows from (2.10) by writing the above sum as

$$e^{ik_r t} \left(p_r(t) + \sum_{j \neq r} e^{i(k_j - k_r)t} p_j(t) \right).$$

Note that in the sum

$$\sum_{V/U} e^{is\xi_0(tH_0)} \sum_{\sigma\in U} P_{s\sigma}(tH_0) \tag{2.12}$$

all the terms $s\xi_0$ are different ($s_1, s_2 \in V$ with $s_1\xi_0 = s_2\xi_0$ implies $s_2^{-1}s_1 \in U$). Thus we can choose $H_0 \in \mathfrak{a}^+$ such that all $s\xi_0(H_0)$ are different.

We shall now show that one of the polynomial in (2.12), namely the one for $s = e$,

$$\sum_{\sigma\in U} P_\sigma(tH_0) \tag{2.13}$$

is not identically 0. For this note that the highest degree term in P_σ is a constant (independent of σ) times

$$\epsilon(\sigma)\frac{1}{\pi''(\lambda_0)}(\sigma\pi'')(tH_0). \tag{2.14}$$

Now each σ permutes the roots vanishing at A_{λ_0}. Hence $\sigma\pi' = \epsilon'(\sigma)\pi'$ where $\sigma \to \epsilon'(\sigma)$ is a homomorphism of U into $\mathbb{R}$. We now use Lemma 2.2. Since each $s_{\alpha_i} \in U$ maps α_i into $-\alpha_i$ and permutes the other positive roots vanishing at λ_0 we see that $\epsilon'(s_{\alpha_i}) = -1 = \epsilon(s_{\alpha_i})$. Thus by Lemma 2.2 $\epsilon'(\sigma) = \epsilon(\sigma)$ for each $\sigma \in U$. Thus (2.14) reduces to

$$\frac{1}{\pi''}\pi'(tH_0).$$

This shows that the polynomial in (2.13) is not identically 0. In view of (2.11) this shows that the lim sup discussed is $\neq 0$ and Lemma 2.3 established.

□

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